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Abstract

This paper discusses a cubic parametrisation of model consistent expectations in a nonlinear dynamic monetary growth model. The so-called Sidrauski model links money, inflation and consumption growth. Iterative least squares combined with simulation is used to address the alleged impact of inflation on consumption growth. It is shown that an increase in especially long-run inflation variability affects the density functions of both consumption and real money balances.

1 Introduction

One of the most lively debates in macroeconomics is the alleged impact of monetary variables on output. Does money affect output and if so to what extent? Traditional Keynesian economics answers this question with a full confirmation, mostly illustrated in a fixed-price IS/LM-framework. New-Keynesian economics uses more sophisticated micro-founded models to confirm the positive impact of money growth on output. On the other hand new-classical economists deny the influence of monetary variables and assume a dichotomy between the financial and real sphere. Inflation is a pure monetary phenomenon and does not have any real consequences, apart from frustrating investors in the long run.

Until the 1980's the focus of the analysis was merely on the short-run impact of monetary policy, with the exception of the late fifties and early sixties, were seminal contributions by Solow (1956) and Sidrauski (1967) were made on (monetary) growth models. In the 1980's Lucas and Romer initiated a new interest in growth theory. Since that time the interest in the monetary effects changed into the expected effects of inflation on output growth. The more static models were abandoned and monetary growth models came in use.

In the short run inflation is believed to have a positive impact on output growth. Even monetary new-classical models forecast real effects from unexpected inflation. In the medium

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term inflation is expected to be neutral. The Phillips-curve is vertical, inflation and output growth are independent. In the long run some believe inflation, and especially inflation uncertainty, to have a negative impact on growth, because inflation increases risk premia.

Growth models need to cope with expectations, because of their dynamic nature. Model consistent expectations, as proposed by Muth (1961), seem to be the most plausible. Why should economic agents have systematic expectational errors in the long run? Model consistent expectations are theoretically attractive but give rise to serious empirical problems. One of these problems is the implementation in nonlinear stochastic growth models. In this paper we implement model consistent expectations in a nonlinear stochastic monetary growth model in the style of Sidrauski. We show how to parameterize such a model. We follow Den Haan and Marcet (1990). The ultimate goal is to asses the effects of inflation on consumption (and through that on output) and money demand.

The next section discusses the Sidrauski model and its equilibrium properties. Section 3 shows the parameterization method. We apply the method for various types of inflation processes in section 4 and summarize in section 5.

2 The Sidrauski model

The key monetary growth model is the Sidrauski model (Sidrauski, 1967). The model extends the Ramsey (1928) optimal savings model by including money balances. The Ramsey model addresses the problem of the optimal savings plan of an infinitely-lived economic agent (say Robinson Crusoë), who is impatient and has to decide on consuming or saving and investing goods. It appears that the rate of time preference and the marginal productivity of his production technology determine the rate of change in his consumption pattern. In a fully deterministic context the agent can predict his whole consumption-saving behaviour.

Sidrauski has extended the Ramsey model through the introduction of money. The consumer now has to decide on consuming, holding money and investing. It is assumed that agents derive utility from holding real money balances (otherwise no money is used). In equilibrium Sidrauski has shown that the rate of change in consumption does not depend on monetary phenomena. Money is superneutral (money growth does not affect output growth). So the classical dichotomy holds.

Since that time a number of authors has attacked and defended the Sidrauski model. Tobin (1980) has shown that the assumption of infinitely lived agents is a necessary precondition to get the superneutrality. If agents have shorter planning horizons inflation is no longer neutral. Moreover, if inflation is able to lower real interest rates, that is to violate the Fisher hypothesis, real effects are likely to be the consequence of monetary policy changes. If money facilitates the production process, that is the marginal productivity of capital is positively influenced by inflation, neutrality no longer holds. Other authors, like Stockman (1981), have defended the Sidrauski model.

In this paper we start with a functional specification of the Sidrauski model. We use the following additive logistic utility function

$$u_t = \frac{c_t^{1-\tau}}{1-\tau} + \frac{m_t^{1-\tau}}{1-\tau} \tag{1}$$

where $0 < \tau < 1$, u_t represents instantaneous utility, c_t real per capita consumption, m_t real per capita money balances. Money is in the utility function, because it can ease shopping for instance (see McCallum and Goodfriend (1991)). Technology is defined by a simple Cobb-Douglas function:

$$f(k_t, k_{t-1}) = \theta_t k_{t-1}^{\alpha} \tag{2}$$

where k_t represents the per capita capital stock, θ_t technology and α the capital share in income. Note that we assume the lagged capital stock to be productive only. This can be explained by a time-to-build argument: capital becomes only productive after installation and serious testing of the equipment.

We assume that the economic agent has to maximize his utility in an infinitely-lived planning horizon, knowing that she prefers consuming now above consumption in the next period by a rate of time preference of β. We can write the planning problem as:

$$\max \sum_{t=0}^{\infty} \beta^t [\frac{c_t^{1-\tau}}{1-\tau} + \frac{m_t^{1-\tau}}{1-\tau}] \tag{3}$$

subject to the budget constraint:

$$c_t + k_t - \mu k_{t-1} + m_t - m_{t-1} + \pi_t m_{t-1} = \theta_t k_{t-1}^{\alpha} \tag{4}$$

where μ represents one minus the fixed rate of depreciation of capital. The right-hand side represents income from production. Income is spent on consumption c_t, gross investment $(1-\mu)k_t$, the change in real money balances $m_t - m_{t-1}$ and inflation tax $\pi_t m_t$.

Up to this moment the model is deterministic. This leads to exact conditions for equilibrium consumption and money demand depending on marginal productivity of capital, population growth n and the rate of time preference β. In this case money is neutral and does not affect real equilibrium. Model consistent expectations turn to perfect foresight.

Suppose we now introduce uncertainty and stochastics in two respects. First we assume technology to be subject to shocks:

$$\log \theta_t = \rho \log \theta_{t-1} + \epsilon_t \tag{5}$$

where ρ is assumed to be positive and less then or equal to one. The logarithmic specification prevents technology to become negative. Technology is assumed to be path dependent. If a shock increases the level of productivity, a fraction ρ will be transferred into the next period. Improvement of technology is assumed to have permanent effects.

The second source of uncertainty is the inflation process. Here it is crucial to assume that we operate in a closed economy, where we assume that the full uncertainty with respect to inflation originates from shocks in money supply, fiscal policy changes, wage policies, etc. We assume that inflation can be modelled using an AR(1)-process:

$$\pi_t = \gamma \pi_{t-1} + \eta_t \tag{6}$$

A sudden unexpected increase in money growth is believed to have a permanent effect on inflation. Sudden changes in inflation affect the inflation tax on money holdings. For instance an increase in inflation might erode the real value of cash balances, favoring consumption and by that growth.

More important is inflation uncertainty. It is an empirical question how to assess inflationary uncertainty. A rough approximation is inflation variability: not the level of inflation is harmfull for real decisions, but the variability. It is known that variability of inflation is correlated with the level of inflation, as prices follow at least first-order nonstationary processes. In our empirical model we experiment with various assumptions on η_t's standard deviation. If inflation variability increases growth should be harmed. Low γ's will increase the long-run variability, so we expect more growth problems with lower γ's.

We can now compute consumption and real money demand in equilibrium:

$$\begin{aligned} c_t^{-\tau} &= \beta E_t[\alpha\theta_{t+1}k_t^{\alpha-1} + \mu](c_{t+1}^{-\tau} + m_{t+1}^{-\tau}) - \\ &\quad (-\theta_t k_{t-1}^{\alpha} - c_t - k_t + \mu k_{t-1} + m_{t-1} - \pi_t m_{t-1})^{-\tau} \qquad (7) \\ m_t^{-\tau} &= \beta E_t[\alpha\theta_{t+1}k_t^{\alpha-1} + \mu](c_{t+1}^{-\tau} + m_{t+1}^{-\tau}) - \\ &\quad (-\theta_t k_{t-1}^{\alpha} - k_t + \mu k_{t-1} - m_t + m_{t-1} - \pi_t m_{t-1})^{-\tau} \qquad (8) \end{aligned}$$

Full equilibrium is described by the two demand equations, the budget restriction and the two stochastic processes (furthermore we assume the no-Ponzi game condition). The model is nonlinear and includes expectations. We assume the expectations to be model consistent: in the long run no systematic expectational errors can be made. Conditional on the information on c_{t-1}, k_{t-1}, m_{t-1}, π_t and θ_t predictions are made for c_t, k_t and m_t. These values are used in expectations processes, which affect current decisions again.

The model explicitly addresses the impact of inflation on growth, as we can vary γ and the standard deviation of η. Through that we are able to circumvent the dichotomy. Inflation uncertainty might decrease real growth, as agents have rational expectations and dislike a loss of real money balances. So we proceed by filling the model with data. In order to quantify the expectational process we have to parameterize the expectations. This stepis not trivial and we elborate on this theme in the next section.

3 Parametrisation of model consistent expectations

Empirical implementation of the model gives rise to solving at least two serious problems. First of all the demand equations are not explicit in c_t and m_t. So we have to parameterize the demand equations. Secondly, it is unknown how the explicit expectations conditions look like. So we need a parameterization of the expectation processes. In this section we closely follow the procedure proposed by Den Haan and Marcet (1990), who parameterize a non-monetary growth model with consistent expectations. Den Haan and Marcet (1990) use the current state variables only in their parameterization. In what follows we include c_{t-1} and m_{t-1} in some of the experiments, as we have an a priori belief that including the lagged variables might assure quicker convergence in our iterative procedure (see hereafter).

We parameterize the right-hand side of the two equilibrium conditions as follows:

$$
\begin{aligned}
c_t^{-\tau} &= \psi_c(k_{t-1}, m_{t-1}, c_{t-1}, \theta_t, \pi_t; \delta, e_t^c) && (9)\\
m_t^{-\tau} &= \psi_m(k_{t-1}, m_{t-1}, c_{t-1}, \theta_t, \pi_t; \phi, e_t^m) && (10)
\end{aligned}
$$

where we assume both ψ_c and ψ_m to be exponential functions, which are linear in the lagged variables and cubic in the joint distribution of the stochastic components with parameter vectors δ and ϕ respectively. The residuals e_t^c and e_t^m are logarithmic. We can write the equations as follows:

$$
\begin{aligned}
\log c_t^{-\tau} &= \delta_1 + \delta_2 k_{t-1} + \delta_3 m_{t-1} + \delta_4 c_{t-1} + \delta_5 \log\theta + \delta_6 \pi_t + \delta_7 \theta_t \pi_t + \\
&\quad \delta_8 (\log\theta_t)^2 + \delta_9 (\pi_t)^2 + e_t^c && (11)\\
\log m_t^{-\tau} &= \phi_1 + \phi_2 k_{t-1} + \phi_3 m_{t-1} + \phi_4 c_{t-1} + \phi_5 \log\theta + \phi_6 \pi_t + \phi_7 \theta_t \pi_t + \\
&\quad \phi_8 (\log\theta_t)^2 + \phi_9 (\pi_t)^2 + e_t^m && (12)
\end{aligned}
$$

where δ_i is the i-th parameter element of vector δ (similar for ϕ_i and ϕ). The parameterization of the expectation processes runs as follows. First we postulate a stochastic process for θ_t and π_t. We simulate series for both variables. These series are used throughout the parameterization process. Initial values for c, m, k, δ and ϕ are chosen. We use the last pair of equations above to calculate series for c and m, given the initial parameter vectors δ_0 and ϕ_0. With these series we make new series for c and m using (7) and (8). Given the fact that agents have rational expectations the old and newly computed series for c and m need to converge if the right parameters are chosen. Next we run a linear least squares regression of $\log c_t^{-\tau}$ and $\log m_t^{-\tau}$ (with c_t and m_t generated by (7) and (8)) on ψ_c and ψ_m to obtain new values for the parameters, δ_1^1 and ϕ_1^1. We go on generating new series for c, m and k until convergence in the parameter vectors δ and ϕ occurs. The parameter vector sequences δ_0,δ_1^1,.... and ϕ_0,ϕ_1^1,.... are given by the following iterative scheme.

$$
\begin{aligned}
\delta_i^i &= (1-\lambda)\delta_{i-1}^{i-1} + \lambda\hat{\delta}_{i-1}^{i-1},\\
\phi_i &= (1-\lambda)\phi_{i-1}^{i-1} + \lambda\hat{\phi}_{i-1}^{i-1}, \qquad\qquad i = 1, 2, ...
\end{aligned}
$$

where $\hat{\delta}_i$ and $\hat{\phi}_i$ are the estimated parameter vectors of the LS regression of the ith iteration, and $\lambda \in (0,1]$.

So an iterative estimation-simulation procedure is used to assure model consistent expectations in a loglinearized growth model. In the next section we address the impact of inflation in such a model.

δ_i	ϕ_i	c_0	m_0	τ	α	β	$1-\mu$
0.1	0.1	1.0	1.0	0.5	0.3	0.95	0.08

Table 4.1 *Values of the initial and constant parameters.*

4 Inflation

In this section we come to the core problem of the paper: the impact on inflation on growth. We run various inflation processes through the a priori (dichotomized in the deterministic version of the) model. Through that we get insight into the stochastic properties of such a complicated growth model.

Table 4.1 lists the parameter and variable assumptions we made. The capital share in production α is assumed to equal 0.3. The rate of time preference $\beta = 0.95$ indicates a discounting of a little over 5 per cent. The parameter τ in the logistic utility function is assumed to be 0.5. The rate of capital depreciation μ is 0.08. Initial values for c and m are set to unity. Initial guesses of the elements of the parameter vectors δ and ϕ are 0.1.

We start with the model including the lagged values c_{t-1} and m_{t-1} in the parametrisation, because we expect quicker covergence in our iterative procedure. Figure 4.1 shows distributions of consumption and money of realizations of both pairs of parameterization functions with inflation process (6), where γ=0.7 and the sample variance of η equals 0.005. The figures show the empirical density functions of consumption and money in a thousand parameterizations experiments. Real money balances erode with inflation. The median value reaches 0.8; the range reaches from 0.4 to unity. Also consumption is affected significantly and reaches a median value below unity. The spread is from 0.5 to 1.5. It appears however that the choice of ψ_c and ψ_m affects the outcomes of the convergence process substantially, such that convergence is not even guaranteed. Moreover, the results appear to be very sensitive to the simulated series of θ and π. The parameter estimates vary a lot across various simulation experiments. Even the signs of the parameter estimates are not robust. This is, not surprisingly, especially the case for the parameters of the productivity shocks and inflation. Trying to decrease this sensitivity we proceed with parameterization functions from which $\log c_{t-1}$ and $\log m_{t-1}$ are excluded again. As in Den Haan and Marcet, the functions now only contain state variables. This reduces the sensitivity of the estimated-simulated parameters, although the values and the signs of the parameters still vary somewhat.

So we use the following set of equations:

$$\begin{aligned}
\log(c_t^{-\tau}) &= \delta_1 + \delta_2 \log k_{t-1} + \delta_3 \log\theta_t + \delta_4\pi_t + \\
&\quad \delta_5 \log\theta_t\pi_t + \delta_6(\log\theta_t)^2 + \delta_7(\pi_t)^2 + e_t^c \qquad (13)\\
\log(m_t^{-\tau}) &= \phi_1 + \phi_2 \log k_{t-1} + \phi_3 \log\theta_t + \phi_4\pi_t + \\
&\quad \phi_5 \log\theta_t\pi_t + \phi_6(\log\theta_t)^2 + \phi_7(\pi_t)^2 + e_t^m. \qquad (14)
\end{aligned}$$

In the remainder of our experiments we used four different assumptions regarding the statistical process of inflation to investigate the impact of inflation on consumption and money. The four

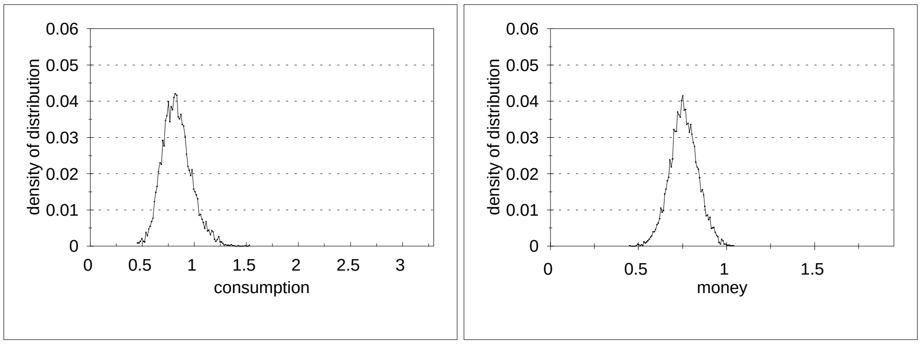

Figure 4.1 *Cumulative empirical density functions of consumption and money. The parameterization function contains c_{t-1} and m_{t-1}. The process of inflation is given by* $\pi_t = 0.7\pi_{t-1} + \varepsilon_t$, $\varepsilon_t \sim \mathcal{N}(0; 0.005)$.

cases are

$$\begin{array}{rrrl} (a) & \pi_t = & 0.7\pi_{t-1} + \varepsilon_t, & \varepsilon_t \sim \mathcal{N}(0; 0.005) \\ (b) & \pi_t = & 0.7\pi_{t-1} + \varepsilon_t, & \varepsilon_t \sim \mathcal{N}(0; 0.01) \\ (c) & \pi_t = & 0.4\pi_{t-1} + \varepsilon_t, & \varepsilon_t \sim \mathcal{N}(0; 0.005) \\ (d) & \pi_t = & 0.5\pi_{t-1} + 0.5\pi_{t-2} + \varepsilon_t, & \varepsilon_t \sim \mathcal{N}(0; 0.005). \end{array}$$

The first three processes reach sensible long-run stochastic equilibria. The first two processes amplify the short-run spread by 3.3, the third by 1.7 (as can be seen from the long-run steady-state solution). The fourth process has no long-run equilibrium.

Figures 4.2 to 4.5 show results of the parameterizing experiments. Again the figures show cumulative density functions for both money and consumption respectively. Figure 4.2 shows the results, similar to 4.1, differing in only one respect: the exclusion of the lagged endogenous variables in the parameterization. The median values do not differ substantially from each other. There is less mass in the tails however. Again nonneutrality rules the model.

The next figures show the same results for inflation processes (b) to (d). Figure 4.3 shows the effects of more inflation uncertainty. Again the median values are not affected, but the mass in the tails changes. More uncertainty increases the chance on extreme values for both c and m. This is what should be expected if inflation uncertainty increases. This result is again ullustrated in figure 4.4, where the long-run inflation variance is smaller again. One can see that more mass is allocated towards median values.

The most extreme pictures are given by the inflation process that has no sensible long-run properties: number (d). One can see now that the median value for consumption drops to 0.6 and that the spread reaches from 0.1 to 3.2. For real money balances there is a large spread of values that have substantial mass: from 0.25 to 1. The distributions have no standard properties however any longer.

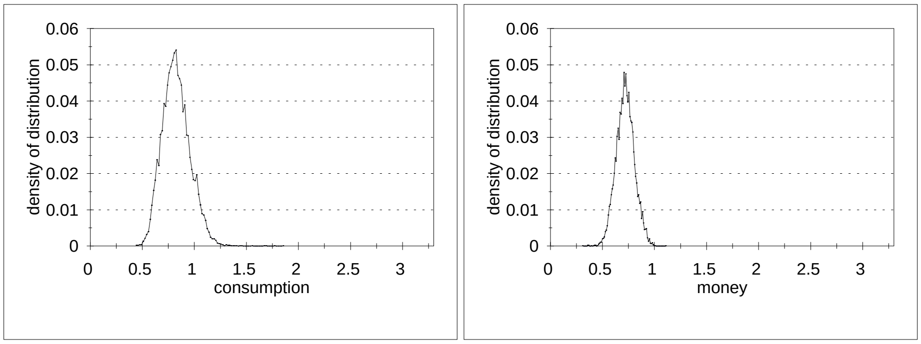

Figure 4.2 *Cumulative empirical density functions of consumption and money. The process of inflation is given by* $\pi_t = 0.7\pi_{t-1} + \varepsilon_t$, $\varepsilon_t \sim \mathcal{N}(0; 0.005)$.

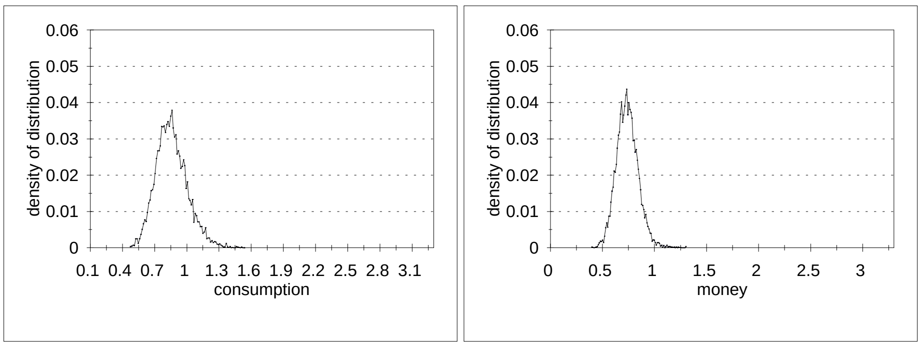

Figure 4.3 *Cumulative empirical density functions of consumption and money. The process of inflation is given by* $\pi_t = 0.7\pi_{t-1} + \varepsilon_t$, $\varepsilon_t \sim \mathcal{N}(0; 0.01)$.

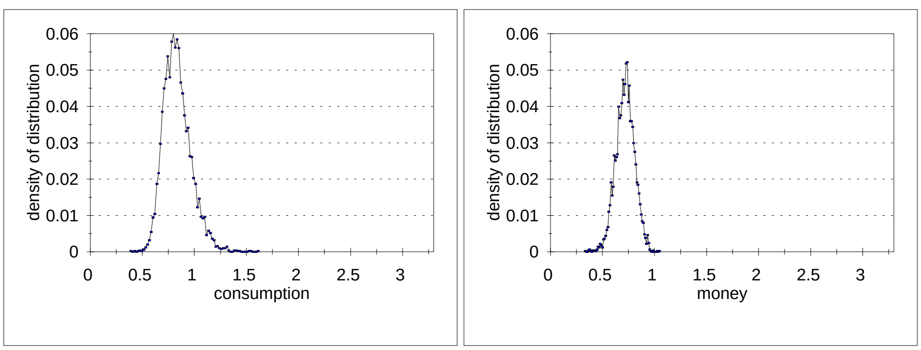

Figure 4.4 *Cumulative empirical density functions of consumption and money. The process of inflation is given by* $\pi_t = 0.4\pi_{t-1} + \varepsilon_t$, $\varepsilon_t \sim \mathcal{N}(0; 0.005)$.

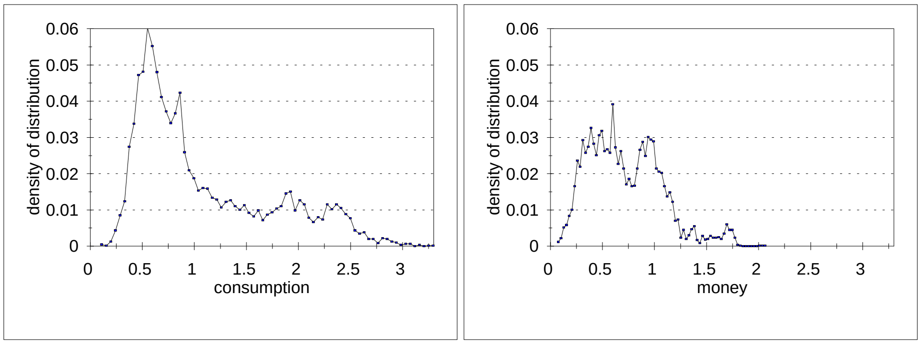

Figure 4.5 *Cumulative empirical density functions of consumption and money. The process of inflation is given by* $\pi_t = 0.5\pi_{t-1} + 0.5\pi_{t-2} + \varepsilon_t$, $\varepsilon_t \sim \mathcal{N}(0; 0.005)$.

So comparing the figures 4.2 to 4.5, it can be noted that the last inflation process, an AR(2) process, gives the largest spread. Using inflation process (c) a lot of trials were needed to get convergence of the parameters. The experiments show that the stochastics of inflation affect the real sphere, even in a neutral environment, through the impact on the formation of expectations.

5 Summary and conclusions

Nonlinear stochastic growth models including model consistent expectations give rise to some empirical problems. In this paper we show how expectations can be parameterized. As an example we use the Sidrauski monetary growth model. This model enables us to study the effects of inflation and inflation variability on consumption growth.

We encounter a number of problems. The first is the choice of the functional parametrisation. Here we show a simple representation. The second problem is to achieve convergence in the iterative estimation-simulation process (we had problems for instance in the case of inflation process (c)). It took a lot of trials to get an amount of completed parameterizations. A point for further investigation could be to find the source and nature of this sensitiveness. A third problem in our specific example is the explanation of some of the outcomes. We find almost likely real effects of inflation in a model that assumes a dichotomy. The nature of the model apparently is highly sensitive for stochastic elements.

A major conclusion is that inflation variability increases the chances of extreme growth paths. This effect is commonly known in the literature (see for instance Orphanides and Solow (1990). This implies that a major source of breaking the neo-classical dichotomy is to be found in the stochastic properties of inflation. It is shown that long-run variablity in particular affects the distribution of consumption patterns.

A possible extension of this line of research is a change in the theoretical growth model. The production function is now given by $f(k_t, k_{t-1}) = \theta_t k_{t-1}^{\alpha}$. One could imagine that money balances are also a factor of interest in the production function, as internal liquidity (retained earnings) is a major source of financing investment. So implementing a production function $f(k, m)$ could be a next step in investigating the implications of inflation in a growth model.

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